UDC 534.01

INVESTIGATION OF OSCILLATIONS IN NONLINEAR MECHANICAL SYSTEMS OF ARBITRARY ORDER

PMM Vol. 42, № 1, 1978, pp. 66-75

V.T. GRUMONDZ
(Moscow)

(Received April 8, 1977)

The problem of existence and stability of steady oscillations of a mechanical system of arbitrary order with the right-hand sides of its equations represented by series in powers of some, generally speaking, not necessarily small parameter is solved. When that parameter vanishes, the system becomes linear whose characteristic equation has a pair of pure imaginary roots with the remaining roots having negative real parts. One of the methods of oscillation investigation of systems of such form is that of Kamenkov [1] which has a number of positive features. His method was essentially developed for second order systems; for systems of arbitrary order a method is indicated in [1] for the derivation of periodic solutions in the form of series in powers of a parameter. Another system of investigation of nonlinear systems of arbitrary order is proposed here. It is based on and is a development of Kamenkov's method [1]. Sufficient conditions of steadiness and stability of oscillations within a range are obtained, and a theorem which makes possible to estimate the extent of that region and, also, the limit value μ_0 of parameter μ such that for all $\mu < \mu_0$ the steadiness and stability conditions are maintained. An example is considered.

1. Let us consider a mechanical system whose behavior is defined by the following system of differential equations:

$$x^{\cdot} = -\lambda y + \mu X_1 + \mu^2 X_2 + \ldots, \quad y^{\cdot} = \lambda x + \mu Y_1 + \mu^2 Y_2 + \ldots \tag{1.1}$$

$$x_j^{\cdot} = \sum_{k=1}^{n} p_{jk} x_k + \mu X_{j1} + \mu^2 X_{j2} + \ldots, \quad j = 1, 2, \ldots, n$$

where n is any positive integer and μ is a positive parameter.

We make the following assumptions about (1.1).

1°. The right-hand sides of (1.1) represent absolutely convergent series in the investigated range of variation of variables x, y, x_j and of parameter μ.

2°. $X_l, \ Y_l, \ X_{jl} \ (j = 1, 2, \ldots, n; \ l = 1, 2, \ldots)$ denote sums of forms relative to variables $x, y, x_1, \ldots, x_n$ of any finite order $\nu_{xl}, \nu_{yl}, \nu_{jl}$ with constant coefficients $a_*^{(l)}, b_*^{(l)}, c_*$, respectively, so that

$$X_l(x, y, x_1, \ldots, x_n) = \sum_{\substack{k_x+k_y+k_1+\ldots+k_n = m_{xl} \\ k_x+k_y \geqslant 1}}^{\nu_{xl}} a_*^{(l)} x^{k_x} y^{k_y} x_1^{k_1} \ldots x_n^{k_n} \tag{1.2}$$

$$l = 1, 2, \ldots$$

and so on; the lower powers of forms $m_{xl}, m_{yl}, m_{jl} \geqslant 1$, and the asterisk $*$ denotes a set of indices $(k_x, k_y, k_1, \ldots, k_n)$.

3°. Roots of the characteristic polynomial $D(\varkappa) = | p_{ij} - \delta_{ij}\varkappa |$ of the adjoint system, i. e. of the system consisting of equations of system (1.1) except the first two, satisfy the condition $\operatorname{Re} \varkappa_j < 0$ $(j = 1, 2, \ldots, n)$ and are of the form $\varkappa_k = g_k + ih_k$, $\varkappa_{k+\alpha} = g_k - ih_k$, $\varkappa_s = d_s$; $g_k < 0, d_s < 0, k = 1, 2, \ldots, \alpha; s = 2\alpha + 1, \ldots, n$, with all $\varkappa_j$ $(j = 1, 2, \ldots, n)$ different. (If a system of $(n+2)$-nd order with a pair of pure imaginary roots and n roots with negative real parts is specified in a general form, we shall assume that the reduction to form (1.1) by means of known linear transformation methods [2] has been already effected).

4°. The right-hand sides of the system, i. e. the system consisting of the first two equations of system (1.1), vanish when $x = y = 0$.

Note that when (1.1) does not satisfy the last condition, then in any case with the supplementary assumption that $m_{xl}, m_{yl}, m_{jl} \geqslant 2$ it is possible to reduce it to the required form using the appropriate transformation [2] which is feasible on the basis of the theorem in Sect. 30 of [2] when condition 3° is satisfied.

Using the reasoning in [3] we transform system (1.1) so that the adjoint system assumes the canonical form $w_j^{\cdot} = \varkappa_j w_j (j = 1, 2, \ldots, n)$ and pass in the transformed system to real variables $z_1, \ldots, z_n$. As the result variables $x_1, \ldots, x_n$ and $z_1, \ldots, z_n$ are related by the following formulas:

$$x_j = -2 \sum_{k=1}^{\alpha} \left\{ \operatorname{Re} \left[\frac{H_j(g_k + ih_k)}{D'(g_k + ih_k)} \right] z_k - \operatorname{Im} \left[\frac{H_j(g_k + ih_k)}{D'(g_k + ih_k)} \right] z_{k+\alpha} \right\} - \tag{1.3}$$

$$\sum_{s=2\alpha+1}^{n} \frac{H_j(d_s)}{D'(d_s)} z_s, \quad D'(\varkappa) = \frac{dD(\varkappa)}{d\varkappa}$$

where in virtue of assumption 3° we have $D'(\varkappa_j) \neq 0$. Polynomials $H_j(\varkappa)$ are derived using the determinant $D(\varkappa)$ [3].

We introduce polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ and, then, the new variable of the form

$$r = \rho + \mu \sum_{q=1}^{m_1} \rho^q u_1^{(q)}(\theta), \quad m_1 = \max\{\nu_{x1}, \nu_{y1}\} \tag{1.4}$$

where $u_1^{(1)}(\theta), \ldots, u_1^{(m_1)}(\theta)$ are periodic functions of θ of period 2π, which are defined below. We assume henceforth that parameter μ is sufficiently small for function ρ, defined in terms of r and θ by Eq. (1.4), to be positive definite with respect to any $r > 0$ and all θ belonging to the region of investigation, and so that throughout that region the following inequalities are satisfied:

$$r > 0, \ H = dr(\rho, \theta)/d\rho > 0 \tag{1.5}$$

We define functions $u_1^{(1)}, \ldots, u_1^{(m_1)}$ and the constants $g_1^{(1)}, \ldots, g_1^{(m_1)}$ as follows:

$$u_1^{(1)} = \ldots = u_1^{(m-1)} \equiv 0, \quad m = \min_{l=1,2,\ldots} \{m_{xl}, m_{yl}\}$$

$$u_1^{(q)}(\theta) = \frac{1}{\lambda}\int_0^\theta [R_1^{(1,q)}(\psi) - g_1^{(q)}]\, d\psi, \quad q = m, \ldots, m_1$$

$$g_1^{(1)} = \ldots = g_1^{(m-1)} = 0, \quad g_1^{(q)} = \frac{1}{2\pi}\int_0^{2\pi} R_1^{(1,q)}(\psi)\, d\psi$$

$$R_1^{(1)}(\rho, \theta) = \sum_{q=m}^{m_1} \rho^q R_1^{(1,q)}(\theta)$$

$$R_1(\rho, \theta, z_1, \ldots, z_n) = \sum_{\substack{k_x+k_y+k_1+\ldots+k_n=m \\ k_x+k_y\geqslant 1}}^{m_1} \rho^{k_x+k_y}(A_*^{(l)}\cos\theta +$$
$$B_*^{(l)}\sin\theta)\cos^{k_x}\theta\, \sin^{k_y}\theta z_1^{k_1}, \ldots, z_n^{k_n}$$

where $R_1^{(1)}(\rho, \theta)$ is that part of function $R_1(\rho, \theta, z_1, \ldots, z_n)$ which is independent of $z_1, \ldots, z_n$; and $A_*^{(l)}$ and $B_*^{(l)}$ are constant coefficients obtained by the above transformations. System (1.1) then assumes the form

$$\begin{aligned}
\rho^{\cdot} H &= \mu L_1(\rho) + \mu P_1(\rho, \theta, z_1, \ldots, z_n) + \mu^2 P_2(\rho, \theta, z_1, \ldots, z_n, \mu) \qquad (1.6)\\
\theta^{\cdot} &= \lambda + \mu F_1(\rho, \theta, z_1, \ldots, z_n) + \mu^2 F_2(\rho, \theta, z_1, \ldots, z_n, \mu)\\
z_k^{\cdot} &= g_k z_k - h_k z_{k+\alpha} + \mu Z_{k1} + \mu^2 Z_{k2} + \ldots\\
z_{k+\alpha}^{\cdot} &= h_k z_k + g_k z_{k+\alpha} + \mu Z_{k+\alpha,1} + \mu^2 Z_{k+\alpha,2} + \ldots\\
z_s^{\cdot} &= d_s z_s + \mu Z_{s1} + \mu^2 Z_{s2} + \ldots, \quad k = 1, 2, \ldots, \alpha; \quad s = 2\alpha + 1, \ldots, n
\end{aligned}$$

Owing to the selection of functions $u_1^{(q)}(\theta)$ and constants $g_1^{(q)}$, the expression $L_1(\rho)]$ is the following polynomial with constant coefficients:

$$L_1(\rho) = \sum_{q=m}^{m_1} g_1^{(q)}\rho^q \qquad (1.7)$$

The second term in the right-hand side of the first of Eqs. (1.6) is

$$\mu P_1 = \mu \sum_{k_x+k_y=0}^{m_1-1} \rho^{k_x+k_y} \sum_{k_1+k_2+\ldots+k_n=1}^{m_1-k_x-k_y} (A_*^{(1)}\cos\theta + B_*^{(1)}\sin\theta) \times$$
$$\cos^{k_x}\theta\, \sin^{k_y}\theta z_1^{k_1}, \ldots, z_n^{k_n}$$

Hence the right-hand sides of (1.6) are absolutely convergent series in powers of parameter μ, whose coefficients are forms of finite orders relative to $z_1, \ldots, z_n$ and ρ with coefficients varying with respect to θ with the general period 2π.

2. The presence of noncritical variables $z_1, \ldots, z_n$ in (1.6) considerably complicates the problem as compared to that solved in [1] for second order systems. Unlike in [1] we investigate the system as follows: we simultaneously seek some limit value μ^* of parameter μ and certain restrictions on the initial values of variables $z_1, \ldots, z_n$ and ρ such that for $0 < \mu \leqslant \mu^*$ and when the representing point

$M(t)$ belongs to that region at $t = t_0$, then for all $t > t_0$ it remains in that region, and oscillations of the system are steady in the meaning defined in [1].

Let us assume that the polynomial $L_1(\rho)$ has positive roots $\rho_1, \ldots, \rho_\gamma$ $(\gamma \leqslant m_1 - m)$ and let among these some root ρ_j be of odd multiplicity s. We use the notation $L_1^{(s)}(\rho_j) = (d^{(s)}L_1(\rho) / d\rho^s)_{\rho=\rho_j}$ and select some arbitrary positive numbers ε_1 and ε_2 which satisfy the inequalities

$$0 < \varepsilon_1 < \rho_{j+1} - \rho_j, \quad 0 < \varepsilon_2 < \rho_j - \rho_{j-1} \tag{2.1}$$

In the space of coordinates $\rho, \theta, z_1, \ldots, z_n$ we determine the sets

$$N_0(\varepsilon_1, \varepsilon_2, A_0) = G_0 \cap K_{01} \cap K_{02}$$
$$\Gamma(\varepsilon_1, \varepsilon_2, A_0) = G' \cup K_1' \cup K_2'$$
$$N(\varepsilon_1, \varepsilon_2, A_0) = G \cap K_1 \cap K_2$$

where A_0 is some positive number and

$$G_0: |z|^2 < A_0^2, \quad 0 \leqslant \theta < 2\pi, \quad \rho > 0$$
$$K_{01}: z_k \text{ — are any } \; 0 \leqslant \theta < 2\pi, \quad 0 < \rho < \rho_\alpha$$
$$K_{02}: z_k \text{ — are any } \; 0 \leqslant \theta < 2\pi, \quad \rho_\beta < \rho$$
$$G': |z|^2 = A_0^2, \quad 0 \leqslant \theta < 2\pi, \quad \rho_\beta \leqslant \rho \leqslant \rho_\alpha$$
$$K_1': |z|^2 \leqslant A_0^2, \quad 0 \leqslant \theta < 2\pi, \quad \rho = \rho_\alpha$$
$$K_2': |z|^2 \leqslant A_0^2, \quad 0 \leqslant \theta < 2\pi, \quad \rho = \rho_\beta$$
$$G: |z|^2 \leqslant A_0^2, \quad 0 \leqslant \theta < 2\pi, \quad \rho > 0$$
$$K_1: z_k \text{ — are any } \; 0 \leqslant \theta < 2\pi, \quad 0 < \rho \leqslant \rho_\alpha$$
$$K_2: z_k \text{ — are any } \; 0 \leqslant \theta < 2\pi, \quad \rho_\beta \leqslant \rho$$
$$\rho_\alpha = \rho_j + \varepsilon_1, \quad \rho_\beta = \rho_j - \varepsilon_2, \quad |z|^2 = z_1^2 + \ldots + z_n^2$$

For $n = 1$ region N represents a segment of a ring-form cylinder in a three-dimensional space.

Let us consider the Liapunov functions $V_1 = \rho$ and $V_2 = {}^1/_2 |z|^2$ that are positive definite with respect to a part of variables.

Lemma 1. $N = N_0 \cup \Gamma$, where Γ is the complete boundary of region N_0. Proof of this lemma is obvious.

Lemma 2. Let the system of differential equations

$$\rho^\cdot = F(\rho, \theta, z_1, \ldots, z_n), \quad \theta^\cdot = \Phi(\rho, \theta, z_1, \ldots, z_n) \tag{2.2}$$
$$z_k^\cdot = Z_k(\rho, \theta, z_1, \ldots, z_n), \qquad k = 1, 2, \ldots, n$$

where functions F, Φ, and Z_k $(k = 1, 2, \ldots, n)$ and their partial derivatives are continuous, satisfy the conditions: $V_2^\cdot < 0$ for all $(\rho, \theta, z_1, \ldots, z_n) \in G'$, $V_1^\cdot < 0$ for all $(\rho, \theta, z_1, \ldots, z_n) \in K_1'$, and $V_1^\cdot > 0$ for all $(\rho, \theta, z_1, \ldots, z_n) \in K_2'$, where $V_1^\cdot$ and $V_2^\cdot$ are derivatives of functions V_1 and V_2 by virtue of system (2.2). Then any representing point $M(t)$ with coordinates $(\rho, \theta, z_1, \ldots, z_n)$ that satisfies the condition $M(t_0) \in N$ for $t = t_0$, satisfies the condition $M(t) \in N_0$ for all $t > t_0$.

Proof. We prove the lemma by proving the following two statements: A) If $M(t_0) \in N_0$, then $M(t) \in N_0$ for all $t > t_0$, and B) if $M(t_0) \in \Gamma$, then $M(t) \in N_0$ for all $t > t_0$. Let us prove the statement A) by assuming the contrary: there exists a $T > t_0$ such that $M(T) \in \Gamma$, and that point $M(t)$ reaches Γ at instant $t = T$ for the first time after instant $t = t_0$. According to Lemma 1, Γ is the complete boundary of region N_0, hence only the following cases are possible:

$$1)\ M(T) \in G',\quad 2)\ M(T) \in K_1',\quad 3)\ M(T) \in K_2'$$

Let us consider the first possibility. The continuity of $V_2^{\cdot}$ implies the existence of $t^* \in [t_0, T)$ such that for all $M(t)$, where $t \in [t^*, T]$, the inequality $V_2^{\cdot} < 0$ is satisfied. For $V_2(t)$ along the trajectory of the representing point $M(t)$ we have

$$V_2(t) = V_2(t^*) + \int_{t^*}^{t} V_2^{\cdot} dt \tag{2.3}$$

By assumption $V_2(t^*) < {}^1/_2 A_0^2$, $V_2(T) = {}^1/_2 A_0^2$, and owing to the selection of t^* the derivative $V_2^{\cdot}(t) < 0$ for all $t \in [t^*, T]$, which contradicts equality (2.3) where $t = T$ is assumed. Similar reasoning proves this statement in cases 2 and 3, and also statement B). Note that in proving the latter it is necessary to use equations of the type (2.3) simultaneously for V_1 and V_2.

Theorem 1. Let ρ_j be a positive root of odd multiplicity s of polynomial $L_1(\rho)$ and let $L_1^{(s)}(\rho_j) < 0$. Then there exist such μ^*, A_0^*, ε_1^*, and such ε_1^* and ε_2^* which satisfy conditions (2.1) that for all $\mu < \mu^*$ and all $t \in (t_0, \infty)$ the representing point $M(t) \in N_0(\varepsilon_1^*, \varepsilon_2^*, A_0^*)$, if only $M(t_0) \in N(\varepsilon_1^*, \varepsilon_2^*, A_0^*)$.

Proof. Consider the following inequalities:

$$\begin{gathered} L_1(\rho_\alpha) + P_{10}(\rho_\alpha, A_0) + \mu P_{20}(\rho_\alpha, \mu, A_0) < 0 \\ L_1(\rho_\beta) - P_{10}(\rho_\beta, A_0) - \mu P_{20}(\rho_\beta, \mu, A_0) > 0 \\ g'A_0 + E(\rho_\alpha, \mu, A_0) < 0 \end{gathered} \tag{2.4}$$

where

$$P_{10}(\rho, A_0) = \sum_{k_x+k_y=0}^{m_1-1} \rho^{k_x+k_y} \sum_{k_1+\ldots+k_n=1}^{m_1-k_x-k_y} ([A_*^{(1)}]^2 + [B_*^{(1)}]^2)^{1/2} A_0^{k_1+\ldots+k_n}$$

$$E(\rho_\alpha, \mu, A_0) = S \sum_{l=1}^{\infty} \mu^l \sum_{j=1}^{n} \sum_{\substack{k_x+k_y+k_1+\ldots+k_n=\eta \\ k_x+k_y\geqslant 1}}^{m_l} f^{k_x+k_y} |C_*^{(j,l)}| A_0^{k_1+\ldots+k_n}$$

$$\eta = \min_{\substack{l=1,2,\ldots \\ j=1,2,\ldots,n}} \{m_{jl}\}, \quad m_l = \max\{\nu_{xl}, \nu_{yl}\}$$

$$f = \rho_\alpha + \mu(\sigma_1^{(1)}\rho_\alpha + \ldots + \sigma_1^{(m_1)}\rho_\alpha^{m_1}), \quad g' = \max_{\substack{k=1,2\ldots,a \\ s=2a+1,\ldots,n}} \{g_k, d_s\} < 0$$

$$S = \max_{\substack{k=1,2,\ldots,a \\ j=1,2,\ldots,n \\ s=2a+1,\ldots,n}} \left\{ \left| \operatorname{Re} \frac{D_{ji}(g_k + ih_k)}{H_i(g_k + ih_k)} \right|, \left| \operatorname{Im} \frac{D_{ji}(g_k + ih_k)}{H_i(g_k + ih_k)} \right|, \left| \frac{D_{ji}(d_s)}{H_i(d_s)} \right| \right\}$$

$$|P_1| \leqslant P_{10}, \quad |P_2| \leqslant P_{20}, \quad |u_1^{(q)}(\theta)| \leqslant \sigma_1^{(q)}, \quad \left| \frac{du_1^{(q)}}{d\theta} \right| \leqslant \sigma_{10}^{(q)}$$

$$0 \leqslant \theta < 2\pi, \quad q = m, \ldots, m_1$$

P_{20} is a series in powers of μ whose coefficients are similar to those of P_{10}; $\sigma_1^{(q)}$ and $\sigma_{10}^{(q)}$ are positive numbers which bound functions $|u_1^{(q)}|$ and $|du_1^{(q)}/d\theta|$; D_{ji} is the cofactor of the element in the j-th row and i-th column of determinant $D(\varkappa)$ and defines the coefficients of the inverse transformation of (1.3).

We assume, without loss of generality, that the region of absolute convergence of series indicated in statement 1° in Sect. 1 is such that the series in the left-hand sides of inequalities (2.4) are convergent in some closed bounded region of variables ε_1, ε_2, and A_0 that contains point $\varepsilon_1 = \varepsilon_2 = A_0 = 0$. We select in that region any ε_1^* and ε_2^* that satisfy conditions (2.1). Since $L_1^{(s)}(\rho_j) < 0$, hence $L_1(\rho_\alpha^*) < 0$ and $L_1(\rho_\beta^*) > 0$, where $\rho_\alpha^* = \rho_j + \varepsilon_1^*$, $\rho_\beta^* = \rho_j - \varepsilon_2^*$. The form of the first two of inequalities (2.4) implies that when $L_1(\rho_\alpha^*) < 0$ and $L_1(\rho_\beta^*) > 0$ there exists such numbers $\mu' > 0$ and $A_0^* > 0$ that the first two of that inequality are satisfied for all $\mu < \mu'$ and $A_0 \leqslant A_0^*$. We select now $\mu^* \leqslant \mu'$ such that for all $\mu < \mu^*$ and the chosen $A_0 = A_0^*$ the last of inequalities (2.4) is satisfied. The form of expression $g'A_0 + E$ implies that this can be always achieved, since $g' < 0$ and $A_0 = A_0^*$ has been already chosen. Then conditions (2.4) are satisfied for all $\mu < \mu^*$ and $A_0 = A_0^*$. The first two of these conditions ensure that the inequalities $V_1^{\cdot} < 0$ and $V_1^{\cdot} > 0$ on K_1' and K_2', respectively, are satisfied, while the last of conditions (2.4) ensures the fulfilment of the inequality $V_2^{\cdot} < 0$ on G', where V_1, V_2, K_1', K_2', and G' were defined earlier, and the derivatives are taken on the basis of (1.6). The latter can be ascertained by a direct check. From this in conformity with Lemma 2 follows Theorem 1.

Thus under the stated conditions the investigated system has oscillations that are steady in the meaning of [1] and correspond to the root ρ_j. They represent motions for which the representing point belongs to N when $t = t_0$.

Theorem 2. Let ρ_j be a positive root of odd multiplicity s of the polynomial $L_1(\rho)$. Then if $L_1^{(s)}(\rho_j) < 0$ and all $\mu < \mu^*$ steady oscillations are stable in region $N(\varepsilon_1^*, \varepsilon_2^*, A_0^*)$ and unstable in region $N_1 = G \cap \bar{K}_{02}$; if $L_1^{(s)}(\rho_j) > 0$, there exist positive numbers μ', A_0', and ε_1' and ε_2' that satisfy conditions (2.1) and are such that for all $\mu < \mu'$ the system motion is unstable in region $N_2 = N(\varepsilon_1', \varepsilon_2', A_0')$. (Here $\bar{K}_{02}$ denotes the addition to region K_{02} and stability in a particular region is understood in the meaning of [4]).

Proof. Stability in region N is obvious. To prove instability it is sufficient to change the direction of variation of t along the trajectory of the system representing point, and select for the initial position of the latter point M_0 of region N_1 without the boundary (or N_2 without the boundary) which is reached by the representing point $M_-(t)$ at some fixed $\tau > t_0$ when the direction of t is reversed, and which at $t = t_0$ belongs to the boundary of N_1 (or N_2). It is obvious that, at least, the representing point $M(t)$, whose motion begins at M_0 leaves region N_1 (or N_2) in the finite time interval $\tau - t_0$. The existence of such μ', A_0', ε_1', and ε_2' which ensure the necessary signs of derivatives at the boundaries of region N_2 is implied, as in Theorem 1, by the analysis of the first two of inequalities (2.4) in which, however, the signs of the second and third terms and the signs of the inequalities themselves have been reversed.

Theorem 3. Let the conditions of Theorem 1 be satisfied and let $\mu_1 = \varphi_1(\varepsilon_1)$ and $\mu_2 = \varphi_2(\varepsilon_2)$ be solutions of equations which are obtained for some $A_0^{**} > 0$

and conditions (2.1) from the first two of inequalities (2.4) by substituting the equality symbol for that of inequality. Furthermore, let $\mu_{10} = \max \varphi_1(\varepsilon_1) = \varphi_1(\varepsilon_{10})$, $\mu_{20} = \max \varphi_2(\varepsilon_2) = \varphi_2(\varepsilon_{20})$, where ε_{10} and ε_{20} satisfy conditions (2.1). Then in any case the quantity $\mu^{**} = \min\{\mu_{10}, \mu_{20}, \sup \mu'\}$, where μ' represents such μ for which conditions (1.5) and the last of conditions (2.4) are satisfied for $\varepsilon_1 = \varepsilon_{10}$ and $\varepsilon_2 = \varepsilon_{20}$, can be taken as the limit value of parameter μ which corresponds to A_0^{**}, ε_{10}, ε_{20}, and ρ_j.

The proof of this theorem is directly obtained from the analysis of inequalities (2.4) and (1.5).

Theorems 1 and 2 establish the conditions of existence and stability of steady oscillations and Theorem 3 outlines the method of deriving the values of μ^{**} and parameters A_0^{**}, ε_{10}, and ε_{20} in region N. Obviously, in the general case $\mu^{**} \leqslant \mu^*$.

If system (1.1) is considered as a system of equations of perturbed motion, the above theorems provide a solution of the problem of stability in the critical case in the region of a pair of pure imaginary roots of the characteristic equation, since according to conditions 2° and 4° polynomial $L_1(\rho)$ has always a zero root, and region N contains in that case the coordinate origin. Theorem 3 provides the means for estimating the region of attraction.

All of the above together with Theorems 1—3 is also valid in the case when in the investigated systems functions X_l, Y_l, and X_{jl} $(j = 1, 2, \ldots, n;\ l = 1, 2, \ldots)$ have as their coefficients functions of time that are continuous and uniformly bounded for $t \in (-\infty, \infty)$ and periodic for X_1 and Y_1, while for the remaining X_l, Y_l, X_{jl} not necessarily periodic, and the right-hand sides of the system contain additional terms in the form of absolutely convergent for $t \in (-\infty, \infty)$, series

$$\sum_{l=1}^{\infty} \mu^{l-1} F_{l-1}(t), \qquad \sum_{l=1}^{\infty} \mu^{l-1} \Phi_{l-1}(t), \qquad \sum_{l=1}^{\infty} \mu^{l} \chi_{jl}(t)$$

respectively, where $F_{l-1}(t)$, $\Phi_{l-1}(t)$, $\chi_{jl}(t)$ are also continuous and uniformly bounded functions $t \in (-\infty, \infty)$ and F_0, F_1, Φ_0, Φ_1 are periodic functions of t, provided that all remaining assumptions about the considered system are maintained.

In such case the investigated systems can, in accordance with [1], be reduced to a form in which the functions that play the part of functions $F_0(t)$, $F_1(t)$, $\Phi_0(t)$, $\Phi_1(t)$ in a critical system, vanish, while functions that play the part of X_1 and Y_1 have only coefficients that are independent of time. It is evident that all other consdierations remain unchanged, except, of course, the form of expressions in conditions (2.4) which is affected by such transformation and the presence of additional terms in (1.1).

3. As an example of the application of above results we shall investigate the character of oscillations of some solid body in a stream of fluid and provided with control organs relative to a fixed direction ξ in space. We assume that the buoyancy force is equal to the weight of the body and is applied at the center of gravity of the latter. The body is subjected to hydrodynamic and control forces and moments. The controls are effected by water jets or screw propellers [5] and the body moves in a horizontal plane which is also the plane of the velocity vector $\mathbf{v}_\infty$ of the oncoming fluid stream, $v_\infty = \text{const}$, $\varphi_v = \text{const}$, and $0 \leqslant \varphi_v < 2\pi$, where φ_v is the angle between that vector and the ξ-direction.

The coordinate system $Axyz$ attached to the body has its origin at the center of gravity and the coordinates lie in the body symmetry planes. Let m be the mass of the body, L its length, S the area of its middle section, I_y the moment of inertia with respect to the vertical axis y, λ_{55} be the apparent moment of inertia, and the apparent masses $\lambda_{11} = \lambda_{33}$.

We use the known equations of motion of a body in a horizontal plane in a fluid (see, e.g., [5]), approximate the dependence of hydrodynamic forces and moments on the angle of drift $\beta = \varphi - \varphi_v$ (φ is the angle between the Ax-axis and the ξ-direction) by Legendre polynomials, and introduce dimensionless variables τ, v_x', v_z', φ', and w_y' defined as follows:

$$t = \frac{\sqrt{S}}{v_\infty}\tau, \quad v_x = v_x' \frac{mv_\infty}{m+\lambda_{11}}, \quad v_z = v_z' \frac{mv_\infty}{m+\lambda_{11}}$$

$$\varphi = \frac{mS}{I_y+\lambda_{55}}\varphi', \quad \omega_y = \omega_y' \frac{mv_\infty\sqrt{S}}{I_y+\lambda_{55}}$$

We specify in the general case the following structure of the dimensionless control forces and moments:

$$F_x = -c_{x0}(\varphi_v)\frac{\rho S^{3/2}}{2m} + l_{x1}v_x' + l_{x2}v_z' + \mu\left[n_{x1}v_x'^3 + n_{x2}v_z'^3\right]$$

$$F_z = -c_{z0}(\varphi_v)\frac{\rho S^{3/2}}{2m} + l_{z1}v_x' + l_{z2}v_z' + \mu\left[n_{z1}v_x'^3 + n_{z2}v_z'^3\right]$$

$$M_y = -m_{y0}(\varphi_v)\rho\frac{SL}{2m} + k_\varphi\varphi' + \mu\left[n_\varphi\varphi'^3 + n_\omega\omega'^3\right]$$

where $c_{x0}(\varphi_v)$, $c_{z0}(\varphi_v)$, and $m_{y0}(\varphi_v)$ are polynomials that are functions of polynomials which approximate hydrodynamic characteristics, and introduce the substitution $\varphi' = -|k_\varphi|^{-1/2}\varphi''$. Then, assuming that

$$k_\varphi < 0, \quad l_{x1} + l_{z2} < 0, \quad (l_{z1} + l_{z2})^2 + 4(l_{x2}l_{z1} - l_{x1}l_{z2}) < 0$$

we obtain, as can be readily verified, a system of equations of the form (1,1) in which the characteristic equation has a pair of pure imaginary roots $\pm i\lambda$ (because $k_\varphi < 0$) and two complex-conjugate roots with real parts equal $l_{x1} + l_{z2}$ when $\mu = 0$, and

$$x = \varphi'', \quad y = \omega', \quad x_1 = v_x', \quad x_2 = v_z', \quad \lambda = |k_\varphi|^{1/2}, \quad \mu = \frac{mS}{I_y+\lambda_{55}}$$

and among the set of coefficients in the right-hand sides there are thirty nonzero.

To reduce the system to the canonical form we further specify that $l_{z1} + l_{z2} \neq l_{x1} + l_{x2}$. The roots of polynomial $L_1(\rho)$ are of the form

$$\rho_1 = 0, \quad \rho_{2,3} = \pm\left[-\frac{2}{3}\frac{m_{y1}^2(\varphi_v)\rho\sqrt{S}L^2}{A(\varphi_v)\,mn_\omega}\right]^{1/2}$$

Functions $m_{y1}(\varphi_v)$ and $A(\varphi_v)$ are polynomials whose coefficients are determined by the coefficients of the approximating polynomials. We thus obtain that when $n_\omega < 0$ and angle φ_v is such that $A(\varphi_v) > 0$ and $m_{y1}(\varphi_v) \neq 0$ then for fairly small μ there exists steady oscillations which are stable with respect to region N and the equilibrium position is unstable. When $n_\omega > 0$ and angle φ_v is such that $A(\varphi_v) < 0$ and $m_{y1}(\varphi_v) \neq 0$, the oscillations in N are unstable; the equilibrium position relative to that region is stable.

Region N in terms of input variables is determined by the following inequalities:

$$\frac{\varphi^2}{A_\alpha^2} + \frac{\omega^2}{B_\alpha^2} \leqslant 1, \quad \frac{\varphi^2}{A_\beta^2} + \frac{\omega^2}{B_\beta^2} \geqslant 1 \tag{3.1}$$

$$(H_{10}v_x + H_{20}v_z)^2 + (H_{11}v_x + H_{21}v_z)^2 \leqslant \left(\frac{A_0 m v_\infty}{m + \lambda_{11}}\right)^2$$

$$A_\alpha = \frac{mS}{\sqrt{|k_\varphi|}(I_y + \lambda_{55})} r_\alpha, \quad B_\alpha = \frac{m v_\infty \sqrt{S}}{I_y + \lambda_{55}} r_\alpha$$

$$H_{10} = \frac{T_2}{\Delta}, \quad H_{20} = -\frac{S_2}{\Delta}, \quad H_{11} = -H_{21} = -\frac{1}{\Delta}$$

$$\Delta = \frac{1}{h_1}(l_{z1} + l_{z2} - l_{x1} - l_{x2})$$

$$T_2 = \frac{1}{h_1}(g_1 + l_{z1} - l_{x1}), \quad S_2 = \frac{1}{h_1}(g_1 + l_{x2} - l_{z2})$$

$$g_1 = {}^1/_2(l_{x1} + l_{z2}), \quad h_1 = {}^1/_2\,|(l_{x1} + l_{z2})^2 + 4(l_{x2}l_{z1} - l_{x1}l_{z2})|^{1/2}$$

$$r_\alpha = (\rho_2 + \varepsilon_{10})\{1 - \mu^{**}[\sigma_1^{(1)} + (\rho_2 + \varepsilon_{10})^2\sigma_1^{(3)}]\}$$

$$r_\beta = (\rho_2 - \varepsilon_{20})\{1 + \mu^{**}[\sigma_1^{(1)} + (\rho_2 - \varepsilon_{20})^2\sigma_1^{(3)}]\}$$

in which A_β and B_β are obtained from A_α and B_α by substituting r_β for r_α. In the case of stable equilibrium position the second of inequalities in (3.1) must be discarded.

Note that in the considered particular case oscillations (if they exist) with respect to φ are not only stable but, also, periodic, as implied by Bendickson's theorem[8]. They are defined the following approximate formula:

$$\varphi(t) = -\mu\frac{\rho_2}{\sqrt{|k_\varphi|}}[1 + \mu(u_1^{(1)}(\lambda\tau) + \rho_2^2 u_1^{(3)}(\lambda\tau))]\cos\lambda\tau$$

$$u_1^{(1)}(\lambda\tau) = \frac{1}{4\lambda}\left[\frac{m_{y1}(\varphi_v)\rho SL}{2m\lambda}(\cos 2\lambda\tau - 1) - \frac{m_{y1}^2(\varphi_v)\rho\sqrt{S}L^2}{2mA(\varphi_v)}\sin 2\lambda\tau\right]$$

$$u_1^{(3)}(\lambda\tau) = \frac{1}{32\lambda}\left[\frac{n_\varphi}{\lambda^3}\cos 4\lambda\tau + n_\omega \sin 4\lambda\tau + 4\frac{n_\varphi}{\lambda^3}\cos 2\lambda\tau - 8n_\omega \sin 2\lambda\tau - 5\frac{n_\varphi}{\lambda^3}\right]$$

Numerical calculations require the availability of specific hydrodynamic characteristics. In this case such calculations make possible the separation of region Φ_{10}

of values of φ_v, in which for $n_\omega > 0$ the equilibrium position $v_x = v_z = \varphi = \omega_y = 0$ is stable relative to N, while oscillations are unstable, and region Φ_{20} of values of φ_v in which for $n_\omega < 0$ the equilibrium position is stable and the oscillations unstable. They also make it possible to determine μ^{**}, A_0^{**}, ε_{10}, ε_{20} and region N that correspond to such cases. For example, the order of magnitude of these quantities for $\varphi_v = 50°\cdot (\varphi_v \in \Phi_{20})$ are as follows: $\mu^{**} \sim 10^{-4}$, $\varepsilon_{10} \sim 1$, $\varepsilon_{20} \sim 1$, $\rho_2 \sim 10$, and $A_0 \sim 10^{-1}$.

The motion of the body, thus, consists of steady oscillations relative to the specified direction ξ and of some, generally speaking, small drift in the horizontal plane; parameters of these motions are estimated by inequalities (3.1).

The author thanks the participants in the seminar on analytical mechanics and its leader, V. V. Rumiantsev, for discussing this paper.

REFERENCES

1. Kamenkov, G. V., Investigation of nonlinear oscillations using Liapunov's functions. Selected Works, Vol. 1, Moscow, "Nauka", 1971.

2. Liapunov, A. M., The general Problem of Motion Stability. Collected Works, Vol. 2, Moscow, Izd. Akad.Nauk SSSR, 1958.

3. Lur'e, A. I., Certain Nonlinear Problems of the Theory of Automatic Control. Moscow-Leningrad, Gostekhizdat, 1951.

4. Kamenkov, G. V., On the stability of motion in cases close to critical. Selected Works, Vol. 1, Moscow, 'Nauka", 1971.

5. Pantov, E. N., Makhin, N. N. and Sheremetov, B. B., Fundamentals of the Theory of Motion of Underwater Apparatuses. Leningrad, "Sudostroenie", 1973.

6. Bendickson, I., On curves defined by differential equations. Uspekhi Matem. Nauk. Vol.9, No.1, 1941.

Translated by J. J. D.